Barker College

Student Number

2020 YEAR 12
TRIAL HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

Staff Involved:

WMD* **ARM**

40 copies

8:30 AM
FRIDAY 14 AUGUST

TOPICS COVERED
- **Complex Numbers I & II**
- **Proof**
- **Integration**
- **Vectors**
- **Mechanics**

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided separately
- In Questions 11-16, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale unless specifically stated

Total marks: 100

Section I – 10 marks (pages 2-6)
- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7-13)
- Attempt Questions 11-16
- Allow about 2 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1–10.

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1. If $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ then a non-zero vector $\underset{\sim}{c}$ such that $\underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot \underset{\sim}{c} = 0$ could be

A. $\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}$

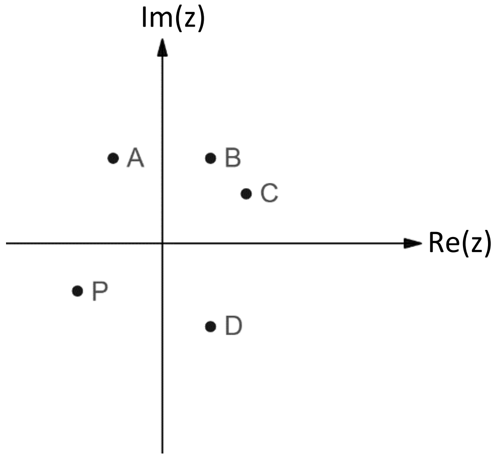
B. $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

C. $\begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$

D. $\begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$

2. A certain complex number $\dfrac{\overline{z}}{i}$ is represented by the point P on the Argand diagram below.

The axes have the same scale.



The complex number z is best represented by

A. A

B. B

C. C

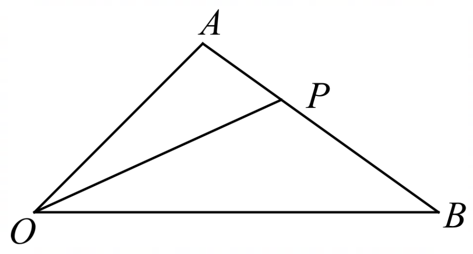
D. D

Section I continues on the next page

Section I continued

3. In this diagram, $\overrightarrow{OA}=\begin{pmatrix}6\\-1\\8\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}-3\\4\\-2\end{pmatrix}$ and $AP:PB=1:2$.

The vector $\overrightarrow{OP}$ is equal to



A. $\frac{1}{3}\begin{pmatrix}0\\7\\4\end{pmatrix}$

B. $\begin{pmatrix}9\\7\\4\end{pmatrix}$

C. $\begin{pmatrix}9\\12\\4\end{pmatrix}$

D. $\frac{1}{3}\begin{pmatrix}9\\2\\14\end{pmatrix}$

4. If $m=a\operatorname{cis}(\theta_1)$ and $n=3\operatorname{cis}(\theta_2)$ and $mn=\frac{1}{2}\operatorname{cis}\left(-\frac{7\pi}{12}\right)$ then a, θ_1 and θ_2 respectively could be

A. $\frac{1}{6}, -\frac{2\pi}{3}, \frac{3\pi}{4}$

B. $6, \frac{2\pi}{3}, \frac{3\pi}{4}$

C. $\frac{1}{6}, \frac{2\pi}{3}, \frac{3\pi}{4}$

D. $6, -\frac{\pi}{3}, \frac{\pi}{4}$

Section I continues on the next page

Section I continued

5. The points R, S and T are collinear.

Given that $\overrightarrow{OR} = \underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{OS} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\overrightarrow{OT} = 3\underset{\sim}{i} + m\underset{\sim}{j} + n\underset{\sim}{k}$, the values of m and n are

A. $m = 3,\ n = -2$

B. $m = -1,\ n = 0$

C. $m = 0,\ n = 1$

D. $m = -3,\ n = 2$

6. Consider the complex numbers $2z$, $-iz$ and $2z - iz$, where $z \neq 0$.

These three complex numbers are plotted in the Argand plane and together with the origin O, they form the vertices of a quadrilateral.

The area of this quadrilateral is

A. $2\left|z^2\right|$

B. $\left|2z\right|$

C. $\left|z^2\right|$

D. $\left|z\right| + \left|2z\right|$

Section I continues on the next page

Section I continued

7. Given the vector $\underset{\sim}{a} = \frac{1}{2}\left(\sqrt{2}\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right)$, then the vector $\underset{\sim}{a}$

A. makes an angle of 120° with the positive *y*-axis and 30° with the positive *z*-axis.

B. makes an angle of 45° with the positive *x*-axis and 120° with the positive *y*-axis.

C. makes an angle of 45° with the positive *x*-axis and 150° with the positive *y*-axis.

D. makes an angle of 135° with the positive *x*-axis and 150° with the positive *y*-axis.

8. The number of distinct roots of the equation $\left(z^2 - 2zi - 1\right)\left(z^2 + 2i\right)\left(z^2 + 2zi + 2\right) = 0$ is

A. 3

B. 4

C. 5

D. 6

Section I continues on the next page

9. The triangle formed by the three points whose position vectors are

$$\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix} \text{ is}$$

A. a scalene triangle

B. a right-angled triangle which is not isosceles

C. an isosceles triangle which is not right-angled

D. a right-angled isosceles triangle

10. By using the substitution $u = \cos x$, or otherwise, it can be shown that $\int \cos^2 x \sin^7 x \, dx =$

A. $-\frac{\cos^3 x}{3} + \frac{3\cos^5 x}{5} - \frac{3\cos^7 x}{7} + \frac{\cos^9 x}{9} + C$

B. $-\cos^3 x + 3\cos^5 x - 3\cos^7 x + \cos^9 x + C$

C. $\frac{\cos^3 x}{3} - \frac{3\cos^5 x}{5} + \frac{3\cos^7 x}{7} - \frac{\cos^9 x}{9} + C$

D. $\cos^3 x - 3\cos^5 x + 3\cos^7 x - \cos^9 x + C$

End of Section I

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z_1 = -1 + 3i$ and $z_2 = 2 - i$.

Find the following, in simplest form:

(i) $z_1 + z_2$ **1**

(ii) $z_1 z_2$ **1**

(iii) $\text{Re}(z_1) - \text{Im}(z_2)$ **1**

(b) (i) Find $\dfrac{d}{dx}\left(\log_e\left(x^2+1\right)\right)$. **1**

(ii) Evaluate $\displaystyle\int_0^{\sqrt{e^2-1}} \frac{4x}{x^2+1}\log_e\left(x^2+1\right)\,dx$, answering in simplified form. **2**

(c) (i) Express $5\cos\theta - 2\sin\theta$ in the form $R\cos(\theta+\alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$. **2**

(ii) Hence, or otherwise, find $\displaystyle\int \frac{dx}{\left(5\cos\frac{x}{2} - 2\sin\frac{x}{2}\right)^2}$. **3**

(d) A complex number z satisfies the inequality $\left|z + 2 - 2\sqrt{3}\,i\right| \le 2$.

(i) Sketch the corresponding region representing all possible values of z. **2**

(ii) Find the set of possible values Arg z can take. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Simplify $(1-i)^6$. **1**

(ii) Solve the equation $z^3 = (1-i)^6$, writing the solutions in the form $x+iy$. **2**

(b) The complex number $3-i$ is denoted by u.

(i) Express $\frac{\overline{u}}{u}$ in the form $x+iy$. **1**

(ii) By considering the argument of $\frac{\overline{u}}{u}$, prove that $\tan^{-1}\left(\frac{3}{4}\right) = 2\tan^{-1}\left(\frac{1}{3}\right)$. **2**

(c) What are the values of real numbers p and q such that $1-i$ is a root of the equation $z^3 + pz + q = 0$? **2**

(d) (i) By using the method of completing the square, or otherwise, solve $z^2 - 2z\cos\theta + 1 = 0$. Give your simplified solutions for z in terms of θ. **2**

(ii) Let α and β be the two solutions found in (i). If P and Q are points on the Argand diagram representing $\alpha^n + \beta^n$ and $\alpha^n - \beta^n$ respectively, show that PQ is of constant length for $n,\ n \in \mathbf{Z}^+$. **2**

(e) Find the coordinates of the point which is nearest to the origin on the line **3**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}.$$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Using the exponential form of a complex number, prove that

$\left(\cos\theta + i\sin\theta\right)^n = \cos(n\theta) + i\sin(n\theta),\ n \in \mathbf{Z}$. **1**

(b) (i) Prove that a possible value of $(-1)^{-i}$ is e^{π}. **1**

(ii) Hence, or otherwise, find an expression for all the possible values of $(-1)^{-i}$. **1**

(c) A particle's motion is defined by the equation $v^2 = 12 + 4x - x^2$, where x is its displacement from the origin in metres and v its velocity in ms^{-1}.
Initially the particle is 6 metres to the right of the origin.

(i) Show that the particle is moving in Simple Harmonic Motion, stating the centre of motion. **2**

(ii) Find the period and the amplitude of the motion. **2**

(iii) The displacement of the particle at any time t is given by the equation **2**

$x = a\sin(nt + \theta) + b.$

Find the values of θ and b, given $0 \le \theta \le 2\pi$.

(d) A particle of mass m kg is fired directly upwards with speed $200\ \text{ms}^{-1}$ in a medium where the resistance is $\frac{1}{10}mv$ newtons when the speed is $v\ \text{ms}^{-1}$. Let $g = 10\ \text{ms}^{-2}$.
Hence, for the upward journey, $\ddot{x} = -\frac{1}{10}(100 + v)$, where x metres is the vertical displacement from the point of projection.

(i) Show that the maximum height attained above the point of projection is $1000(2 - \log_e 3)$ metres. **3**

(ii) Show that the speed $v\ \text{ms}^{-1}$ of the particle on return to its point of projection satisfies the equation $\frac{v}{100} + \ln\left|1 - \frac{v}{100}\right| + (2 - \ln 3) = 0.$ **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, …. **2**

The square numbers are 1, 4, 9, 16, 25, 36, 49, 64, ….

With one exception, if a square number differs from a prime by 1, the prime is bigger.

For example, 16 is a square and 17 is a prime. The prime is bigger. The only exception is prime 3 and square 4.

It is conjectured that for all consecutive numbers that are a prime and a square, except for 3 and 4, the prime is bigger.

Prove the conjecture is true.

(b) Use proof by contradiction to prove that if a, b are integers, then $a^2 - 4b - 3 \neq 0$. **3**
You may wish to consider the case when a is even and the case when a is odd.

(c) A sequence of numbers is given by $T_1 = 6$, $T_2 = 27$ and $T_n = 6T_{n-1} - 9T_{n-2}$ for integers $n \geq 3$. **3**
Use mathematical induction to show that $T_n = (n+1)3^n$ for integers $n \geq 1$.

(d) (i) Use De Moivre's theorem to show that $(1 + i\tan\theta)^5 = \dfrac{\cos 5\theta + i\sin 5\theta}{\cos^5\theta}$. **1**

(ii) Hence find expressions for $\cos 5\theta$ and $\sin 5\theta$ in terms of $\tan\theta$ and $\cos\theta$. **2**

(iii) Show that $\tan 5\theta = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ where $t = \tan\theta$. **1**

(iv) Use the result of (iii) and an appropriate substitution **3**

to show that $\tan\dfrac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$.

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The trajectory of a projectile fired with speed u ms^{-1} at an angle θ to the horizontal, in a medium whose resistance to the projectile's motion is proportional to the projectile's velocity, is represented by the parametric equations

$$x = \frac{u\cos\theta}{k}\left(1 - e^{-kt}\right) \quad \text{and} \quad y = \frac{\left(10 + ku\sin\theta\right)}{k^2}\left(1 - e^{-kt}\right) - \frac{10t}{k},$$

where k is the constant of proportionality of the resistance.

(i) Show the greatest height is reached when $t = \dfrac{1}{k}\log_e\left(\dfrac{10 + ku\sin\theta}{10}\right)$. **2**

(ii) If $k = 0.5$, $u = 40$ and $\theta = 30^\circ$, show that the greatest height is reached when the projectile is at the point $\left(20\sqrt{3},\ 40\left(1 - \log_e 2\right)\right)$. **2**

(b) $I_n = \displaystyle\int_0^1 \frac{1}{\left(1 + x^2\right)^n}\,dx, \ n = 1, 2, 3, \ldots$

(i) Show that $I_{n+1} = \dfrac{2n - 1}{2n} I_n + \dfrac{1}{\left(2^{n+1}\right)n}$. **3**

(ii) Hence evaluate $\displaystyle\int_0^1 \frac{1}{\left(1 + x^2\right)^3}\,dx$. **2**

Question 15 continues on the next page

Question 15 continued

(c) Relative to a fixed origin O, the points A, B and C have position vectors

$$\underset{\sim}{a} = \begin{pmatrix} -1 \\ \frac{4}{3} \\ 7 \end{pmatrix},\ \underset{\sim}{b} = \begin{pmatrix} 4 \\ \frac{4}{3} \\ 2 \end{pmatrix} \text{ and } \underset{\sim}{c} = \begin{pmatrix} 6 \\ \frac{16}{3} \\ 2 \end{pmatrix} \text{ respectively.}$$

(i) Find the cosine of $\angle ABC$. **1**

(ii) Hence find the area of the triangle ABC. **1**

(iii) Use a vector method to find the shortest distance between the point A and the line passing through the points B and C. **2**

Let D be the point such that the quadrilateral $ABCD$ is a kite, where $BC = CD$ and $BA = AD$.

(iv) Find the position vector of the point D. **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use the substitution $t = \tan\frac{x}{2}$ to show $\int_0^{\frac{\pi}{2}} \frac{dx}{2 - \cos x + 2\sin x} = \log_e 2$. **4**

(b) (i) Write down an expression for the sum of the arithmetic series **1**

$1 + 2 + 3 + 4 + ... + (n-1)$.

(ii) Using calculus, or otherwise, show that for $x > 0,\ x > \log_e(1 + x)$. **2**

(iii) Using (i) and (ii), show that $e^{\binom{n}{2}} > n!$ for positive integers $n = 2, 3, 4...$ **3**

(c) Consider the function $f(x) = \sum_{k=1}^{n} \left(\sqrt{a_k}\, x - \frac{1}{\sqrt{a_k}} \right)^2$ where $a_1, a_2, ..., a_n$ are positive real numbers.

(i) By expressing $f(x)$ as a quadratic function of x, show that **3**

$$(a_1 + a_2 + ... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} \right) \geq n^2.$$

(ii) Hence show that $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} \geq \frac{2n}{n+1}$. **2**

End of Question 16

End of Paper

YEAR 12 EXTENSION 2 TRIAL EXAMINATION 2020

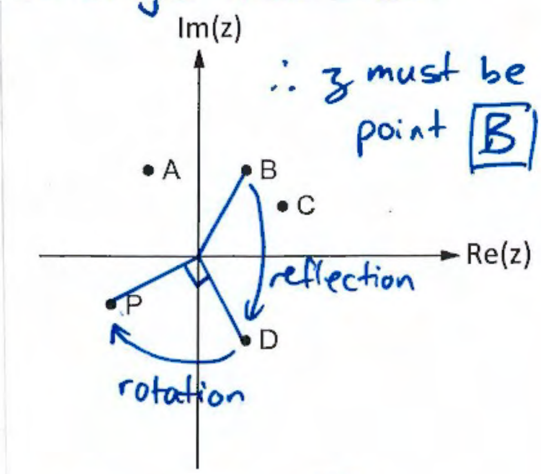
1. Test each answer.

$$\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\3\end{pmatrix} = -2+5-3=0$$

$$\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\3\end{pmatrix} = -1-5+6=0$$

[C]

2. To produce $\frac{\bar{z}}{i}$, z has been reflected in the real axis then rotated $\frac{\pi}{2}$ about the origin clockwise.



$\therefore$ z must be point [B]

3. $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$

$= \overrightarrow{OA} + \frac{1}{3}(\overrightarrow{OB} - \overrightarrow{OA})$

$$= \begin{pmatrix}6\\-1\\8\end{pmatrix} + \frac{1}{3}\left[\begin{pmatrix}-3\\4\\-2\end{pmatrix} - \begin{pmatrix}6\\-1\\8\end{pmatrix}\right]$$

$$= \begin{pmatrix}6\\-1\\8\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-9\\5\\-10\end{pmatrix}$$

$$= \begin{pmatrix}3\\\frac{2}{3}\\\frac{14}{3}\end{pmatrix} = \frac{1}{3}\begin{pmatrix}9\\2\\14\end{pmatrix}$$

[D]

4. $mn = 3a\,\text{cis}(\theta_1+\theta_2)$

$\therefore \frac{1}{2} = 3a$ so $a = \frac{1}{6}$ $\therefore$ A or C

Try $\frac{-2\pi}{3} + \frac{3\pi}{4} = \frac{\pi}{12} \neq \frac{-7\pi}{12}$ (not A)

Try $\frac{2\pi}{3} + \frac{3\pi}{4} = \frac{17\pi}{12}$

$\frac{17\pi}{12} - 2\pi = \frac{-7\pi}{12}$ [C]

5. Since collinear, $\overrightarrow{RS} = \lambda\overrightarrow{ST}$

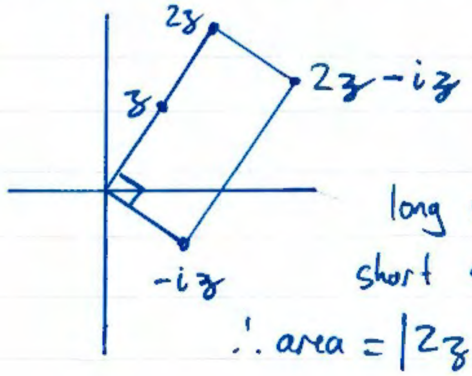
$$\begin{pmatrix}2\\-1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \lambda\left[\begin{pmatrix}3\\m\\n\end{pmatrix} - \begin{pmatrix}2\\-1\\1\end{pmatrix}\right]$$

$$\begin{pmatrix}1\\-2\\1\end{pmatrix} = \lambda\begin{pmatrix}1\\m+1\\n-1\end{pmatrix}$$ so $\lambda = 1$

$-2 = m+1$ $\quad$ $1 = n-1$

$m = -3$ $\quad$ $n = 2$ [D]

6.



long side length $= |2z|$

short side length $= |z|$

$\therefore$ area $= |2z||z| = 2|z^2|$ [A]

7. $\underset{\sim}{a} = \frac{1}{\sqrt{2}}\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$

$\cos\theta_x = \frac{1}{\sqrt{2}}$ $\quad$ $\cos\theta_y = \frac{-1}{2}$ $\quad$ $\cos\theta_z = \frac{1}{2}$

$\theta_x = 45°$ $\quad$ $\theta_y = 120°$ $\quad$ $\theta_z = 60°$

[B]

8. Factorise to:

$$(z-i)^2(z^2+2i)(z^2+2zi+2)=0$$

Degree 6 polynomial with one double root $\Rightarrow$ 5 distinct roots.

[C]

9. Vectors for the three sides:

$$\begin{pmatrix}4\\5\\1\end{pmatrix}-\begin{pmatrix}2\\4\\-1\end{pmatrix}=\begin{pmatrix}2\\1\\2\end{pmatrix},\ \left|\begin{pmatrix}2\\1\\2\end{pmatrix}\right|=3$$

$$\begin{pmatrix}3\\6\\-3\end{pmatrix}-\begin{pmatrix}4\\5\\1\end{pmatrix}=\begin{pmatrix}-1\\1\\-4\end{pmatrix},\ \left|\begin{pmatrix}-1\\1\\-4\end{pmatrix}\right|=3\sqrt{2}$$

$$\begin{pmatrix}2\\4\\-1\end{pmatrix}-\begin{pmatrix}3\\6\\-3\end{pmatrix}=\begin{pmatrix}-1\\-2\\2\end{pmatrix},\ \left|\begin{pmatrix}-1\\-2\\2\end{pmatrix}\right|=3$$

$\therefore$ isosceles

And since $3^2+3^2=(3\sqrt{2})^2$

$\left(\text{or since } \begin{pmatrix}2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}=0\right)$ also

a right angle triangle. [D]

10. $\frac{du}{dx}=-\sin x \Rightarrow dx=\frac{-du}{\sin x}$

$$I=\int u^2(1-u^2)^3\sin x\cdot\frac{-du}{\sin x}$$

$$=-\int u^2(1-3u^2+3u^4-u^6)\,du$$

$$=-\int(u^2-3u^4+3u^6-u^8)\,du$$

$$=\frac{-u^3}{3}+\frac{3u^5}{5}-\frac{3u^7}{7}+\frac{u^9}{9}+C$$

[A]

Question 11.

a) i) $-1+3i+2-i=1+2i$

ii) $(-1+3i)(2-i)=-2+i+6i-3i^2$

$=-2+3+7i$

$=1+7i$

iii) $-1-(-1)=0$

b) i) $\frac{2x}{x^2+1}$

ii) $\int_0^{\sqrt{e^2-1}} 2\cdot\frac{2x}{x^2+1}\cdot\log_e(x^2+1)\,dx$

$$=\left[2\cdot\tfrac{1}{2}\left(\log_e(x^2+1)\right)^2\right]_0^{\sqrt{e^2-1}}$$

$$=\left(\log_e(e^2-1+1)\right)^2-\left(\log_e(0+1)\right)^2$$

$$=\left(\log_e(e^2)\right)^2-0$$

$$=4$$

c) i) $5\cos\theta-2\sin\theta=R\cos\theta\cos\alpha-R\sin\theta\sin\alpha$

$5=R\cos\alpha$ $\quad R=\sqrt{5^2+2^2}$

$2=R\sin\alpha$ $\quad =\sqrt{29}$

$\frac{2}{5}=\tan\alpha \Rightarrow \alpha=\tan^{-1}\frac{2}{5}$

$\therefore 5\cos\theta-2\sin\theta=\sqrt{29}\cos\left(\theta+\tan^{-1}\frac{2}{5}\right)$

ii) $I=\int\frac{dx}{\left(\sqrt{29}\cos\left(\frac{x}{2}+\alpha\right)\right)^2}$ where $\alpha=\tan^{-1}\frac{2}{5}$

$$=\frac{1}{29}\int\sec^2\left(\frac{x}{2}+\alpha\right)dx$$

$$=\frac{1}{29}\cdot2\tan\left(\frac{x}{2}+\alpha\right)+C$$

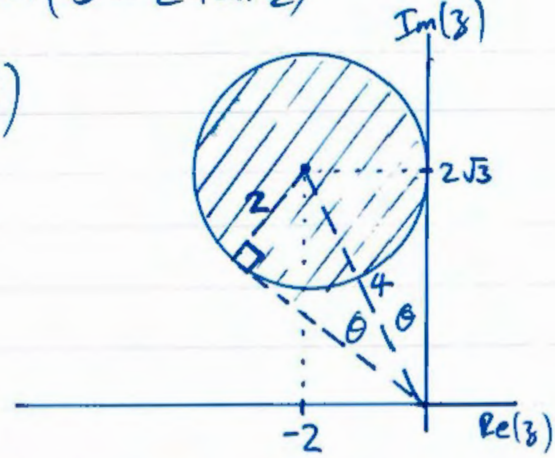
$$=\frac{2}{29}\tan\left(\frac{x}{2}+\tan^{-1}\frac{2}{5}\right)+C$$

or, alternatively:

$$= \frac{2}{29} \frac{\tan\frac{x}{2} + \frac{2}{5}}{1 - \left(\tan\frac{x}{2}\right)\left(\frac{2}{5}\right)}$$

$$= \frac{2(5\tan\frac{x}{2} + 2)}{29(5 - 2\tan\frac{x}{2})}$$

d) i)



Circle, radius 2, centre $-2 + 2\sqrt{3}\,i$

ii) Minimum value for $\mathrm{Arg}(z) = \frac{\pi}{2}$

$|-2+2\sqrt{3}i| = \sqrt{4+(2\sqrt{3})^2} = \sqrt{16} = 4$

so $\sin\theta = \frac{2}{4}$ and $\theta = \frac{\pi}{6}$

Max value for $\mathrm{Arg}(z) = \frac{\pi}{2} + 2\times\frac{\pi}{6} = \frac{5\pi}{6}$

$\therefore \quad \frac{\pi}{2} \le \mathrm{Arg}(z) \le \frac{5\pi}{6}$

Question 12

a) i) $(1-i)^6 = \left(\sqrt{2}\, e^{-\frac{\pi}{4}i}\right)^6$

$= 8e^{-\frac{6\pi}{4}i}$

$= 8e^{\frac{\pi}{2}i} = 8i$

ii) $z^3 = 8i$ let $z = r\,\mathrm{cis}\,\theta$

$r^3 \,\mathrm{cis}\,3\theta = 8\,\mathrm{cis}\left(\frac{\pi}{2} + 2k\pi\right)$

$r^3 = 8 \implies r = 2$

$3\theta = \frac{\pi}{2} + 2k\pi$

$\theta = \frac{\pi(1+4k)}{6}$

and $-\pi < \frac{\pi(1+4k)}{6} \le \pi$

$-6 < 1+4k \le 6$

$\frac{-7}{4} < k \le \frac{5}{4}$

So, for $k = -1, 0, 1$

$\theta = \frac{-3\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

$z = 2\,\mathrm{cis}\left(\frac{-\pi}{2}\right), 2\,\mathrm{cis}\left(\frac{\pi}{6}\right), 2\,\mathrm{cis}\left(\frac{5\pi}{6}\right)$

$= -2i, \sqrt{3}+i, -\sqrt{3}+i$

b) i) $\frac{3+i}{3-i} \times \frac{3+i}{3+i} = \frac{9+6i+i^2}{9+1}$

$= \frac{8+6i}{10}$

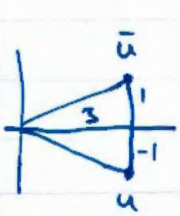
$= \frac{4}{5} + \frac{3}{5}i$

ii)

$\arg\left(\frac{\bar{u}}{u}\right) = \tan^{-1}\left(\frac{3/5}{4/5}\right)$

$= \tan^{-1}\left(\frac{3}{4}\right)$ — ①

But $\arg\left(\frac{\bar{u}}{u}\right) = \arg\bar{u} - \arg u$



$= \tan^{-1}\left(\frac{1}{3}\right) - \tan^{-1}\left(\frac{-1}{3}\right)$

$= \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$= 2\tan^{-1}\left(\frac{1}{3}\right)$ — ②

From ① and ②:

$\tan^{-1}\left(\frac{3}{4}\right) = 2\tan^{-1}\left(\frac{1}{3}\right)$

c) If $1-i$ is a root,

$$(1-i)^3 + p(1-i) + q = 0$$
$$-2-2i + p - pi + q = 0$$
$$(-2+p+q) + (-2-p)i = 0$$
$$-2-p=0 \qquad -2+p+q=0$$
$$p=-2 \qquad -4+q=0$$
$$q=4$$

d) i)

$$z^2 - 2z\cos\theta + \cos^2\theta = -1 + \cos^2\theta$$
$$(z-\cos\theta)^2 = -\sin^2\theta$$
$$z - \cos\theta = \pm i\sin\theta$$
$$z = \cos\theta \pm i\sin\theta$$

ii) $\alpha = \cos\theta + i\sin\theta$

$$\alpha^n = \cos(n\theta) + i\sin(n\theta)$$
$$\beta = \cos\theta - i\sin\theta$$
$$= \cos(-\theta) + i\sin(-\theta)$$
$$\beta^n = \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos(n\theta) - i\sin(n\theta)$$
$$\alpha^n + \beta^n = 2\cos(n\theta)$$
$$\alpha^n - \beta^n = 2i\sin(n\theta)$$

$$PQ = \left|(\alpha^n+\beta^n) - (\alpha^n-\beta^n)\right|$$
$$= \left|2\cos(n\theta) - 2i\sin(n\theta)\right|$$
$$= 2\sqrt{\cos^2(n\theta)+\sin^2(n\theta)}$$
$$= 2\sqrt{1}$$
$$= 2$$

e) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-\lambda \\ 2-3\lambda \\ 2 \end{pmatrix}$

Distance from O to a point on the line is:

$$d = \sqrt{x^2 + y^2 + z^2}$$
$$d^2 = (1-\lambda)^2 + (2-3\lambda)^2 + 2^2$$
$$= 1 - 2\lambda + \lambda^2 + 4 - 12\lambda + 9\lambda^2 + 4$$
$$= 10\lambda^2 - 14\lambda + 9$$

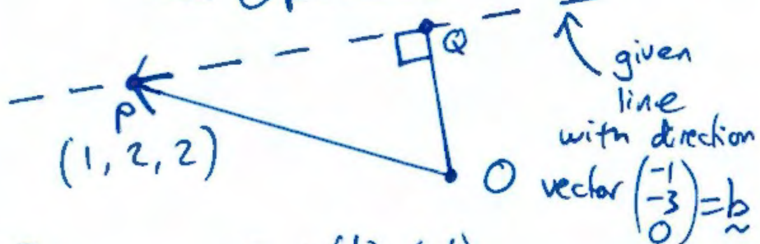
minimum value of d^2 (and hence d) is when $\lambda = \frac{-b}{2a}$

$$= \frac{14}{20} = \frac{7}{10}$$

So, closest point to O is:

$$\left(1-\tfrac{7}{10},\ 2-\tfrac{3\times 7}{10},\ 2\right) = \left(\tfrac{3}{10},\ \tfrac{-1}{10},\ 2\right)$$

— OR —



$$\vec{QP} = \text{proj}_{\underset{\sim}{b}} \vec{OP} = \frac{\begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\-3\\0\end{pmatrix}}{\begin{pmatrix}-1\\-3\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\-3\\0\end{pmatrix}} \begin{pmatrix}-1\\-3\\0\end{pmatrix}$$
$$= \frac{-7}{10}\begin{pmatrix}-1\\-3\\0\end{pmatrix}$$
$$= \begin{pmatrix}\frac{7}{10}\\ \frac{21}{10}\\ 0\end{pmatrix}$$

$$\vec{OQ} = \vec{OP} - \vec{QP} = \begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}\frac{7}{10}\\ \frac{21}{10}\\ 0\end{pmatrix} = \begin{pmatrix}\frac{3}{10}\\ -\frac{1}{10}\\ 2\end{pmatrix}$$

Question 13

a) $\cos\theta + i\sin\theta = e^{i\theta}$

$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$

$= e^{in\theta}$

$= \cos(n\theta) + i\sin(n\theta)$

b) i) Euler's identity:

$e^{i\pi} = -1$

$(e^{i\pi})^{-i} = (-1)^{-i}$

$e^{-i^2\pi} = (-1)^{-i}$

$e^{\pi} = (-1)^{-i}$

ii) Similarly:

$-1 = e^{i(\pi + 2k\pi)}$, $k \in \mathbb{Z}$

$(-1)^{-i} = e^{-i^2(\pi + 2k\pi)}$

$= e^{(2k+1)\pi}$

c) i) $\frac{1}{2}v^2 = 6 + 2x - \frac{1}{2}x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 - x$

$\ddot{x} = -1^2(x-2)$

which is of the form

$\ddot{x} = -n^2(x - c)$

$\therefore$ SHM with centre 2.

ii) min/max x at $v=0$:

$0 = 12 + 4x - x^2$

$0 = (x-6)(x+2)$

$x = -2, 6$

amplitude $= 6 - 2 = 2 - -2 = 4$

period $= \frac{2\pi}{n} = \frac{2\pi}{1} = 2\pi$

(iii) centre of motion, $b = 2$

$x = 4\sin(t + \theta) + 2$

at $t = 0$, $x = 6$

$6 = 4\sin(0 + \theta) + 2$

$1 = \sin\theta$

$\theta = \frac{\pi}{2}$

d) i)

$v\frac{dv}{dx} = \frac{-1}{10}(100 + v)$

$\int_{200}^{0} \frac{v\,dv}{100+v} = \frac{-1}{10}\int_0^h dx$

$\int_{200}^{0} \frac{100 + v - 100}{100 + v}\,dv = \frac{-1}{10}\left[x\right]_0^h$

$\int_{200}^{0} \left(1 - \frac{100}{100+v}\right)dv = \frac{-h}{10}$

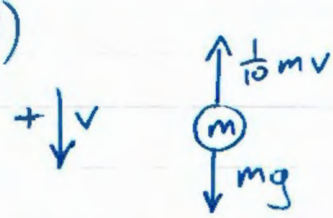
$\left[v - 100\ln(100 + v)\right]_{200}^{0} = \frac{-h}{10}$

$-100\ln 100 - 200 + 100\ln 300 = \frac{-h}{10}$

$-2000 + 1000\ln\left(\frac{300}{100}\right) = -h$

$h = 1000(2 - \ln 3)$

ii)



Let max height be $x = 0$ now. ($v = 0$ there)

$ma = F$

$ma = mg - \frac{1}{10}mv$

$v\frac{dv}{dx} = 10 - \frac{v}{10}$

$$v\frac{dv}{dx} = \frac{100-v}{10}$$

$$\int_0^v \frac{v\,dv}{100-v} = \frac{1}{10}\int_0^{1000(2-\ln 3)} dx$$

$$-\int_0^v \frac{v-100+100}{v-100}\,dv = \frac{1}{10}\left[x\right]_0^{1000(2-\ln 3)}$$

$$\int_0^v \left(1+\frac{100}{v-100}\right) dv = \frac{-1}{10}\left(1000(2-\ln 3)\right)$$

$$\left[v + 100\ln|v-100|\right]_0^v = -100(2-\ln 3)$$

$$v + 100\ln|v-100| - 0 - 100\ln|0-100| + 100(2-\ln 3) = 0$$

$$\frac{v}{100} + \ln\left|\frac{v-100}{100}\right| + (2-\ln 3) = 0$$

$$\frac{v}{100} + \ln\left|\frac{v}{100}-1\right| + (2-\ln 3) = 0$$

$$\frac{v}{100} + \ln\left|1-\frac{v}{100}\right| + (2-\ln 3) = 0$$

Question 14

a) Let the prime be p and the square number be n^2.

For $n^2=1$ and $p=2$, $p>n^2$.

For $n^2=4$ and $p=3$, $p<n^2$ (the exception)

For $n \geq 3$, if $p=n^2-1$ then $p=(n-1)(n+1)$ where, since $n\geq 3$, $n-1>1$ and $n+1>1$ which implies that p is composite.

This is a contradiction so the situation where the prime is one less than the square is impossible. Hence, for $n \geq 3$, $p=n^2+1$.

$\therefore$ The conjecture is true.

b) Assume $a^2-4b-3=0$ $(a, b \in \mathbb{Z})$

so $a^2-4b=3$

If a is even, $a=2n$ $(n\in\mathbb{Z})$

$$(2n)^2-4b=3$$
$$4n^2-4b=3$$
$$2(2n^2-2b)=3$$

which is a contradiction since LHS is even.

If a is odd, $a=2n+1$ $(n\in\mathbb{Z})$

$$(2n+1)^2-4b=3$$
$$4n^2+4n-4b=2$$
$$4(n^2+n-b)=2$$
$$n^2+n-b=\frac{1}{2}$$

which is a contradiction since LHS must be an integer

$\therefore a, b \in \mathbb{Z} \Rightarrow a^2-4b-3 \neq 0$

c) Prove for $n=1$:

$$T_1 = (1+1)3^1 = 2\times 3 = 6 \quad \text{as stated}$$

Prove for $n=2$:

$$T_2 = (2+1)3^2 = 3\times 9 = 27 \quad \text{as stated}$$

Assume true for $n=k$, $k \geq 1$

i.e. $T_k = (k+1)3^k$

Assume true for $n=k+1$, $k \geq 1$

i.e. $T_{k+1} = (k+1+1)3^{k+1}$

$= (k+2)3^{k+1}$

Now prove for $n=k+2$, $k \geq 1$

RTP: $T_{k+2} = (k+3)3^{k+2}$

$T_{k+2} = 6T_{k+2-1} - 9T_{k+2-2}$

$= 6T_{k+1} - 9T_k$

$= 6(k+2)3^{k+1} - 9(k+1)3^k$

$= 2(k+2).3.3^{k+1} - (k+1).3^2.3^k$

$= (2k+4)3^{k+2} - (k+1)3^{k+2}$

$= (2k+4-k-1)3^{k+2}$

$= (k+3)3^{k+2}$

Hence by the principle of mathematical induction, the statement is true $\forall n \in \mathbb{Z}, n \geq 1$.

d) i) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$(\cos\theta[1 + i\tan\theta])^5 = \cos 5\theta + i\sin 5\theta$

$\cos^5\theta(1 + i\tan\theta)^5 = \cos 5\theta + i\sin 5\theta$

$(1 + i\tan\theta)^5 = \dfrac{\cos 5\theta + i\sin 5\theta}{\cos^5\theta}$

ii) From part i)

LHS $= 1^5 + 5i\tan\theta + 10i^2\tan^2\theta + 10i^3\tan^3\theta + 5i^4\tan^4\theta + i^5\tan^5\theta$

$= 1 - 10\tan^2\theta + 5\tan^4\theta + i(5\tan\theta - 10\tan^3\theta + \tan^5\theta)$

RHS $= \dfrac{\cos 5\theta}{\cos^5\theta} + i\dfrac{\sin 5\theta}{\cos^5\theta}$

Equating real/imaginary parts:

$\dfrac{\cos 5\theta}{\cos^5\theta} = 1 - 10\tan^2\theta + 5\tan^4\theta$

$\cos 5\theta = \cos^5\theta(1 - 10\tan^2\theta + 5\tan^4\theta)$

Similarly:

$\sin 5\theta = \cos^5\theta(5\tan\theta - 10\tan^3\theta + \tan^5\theta)$

iii) $\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$

$= \dfrac{\cos^5\theta(5t - 10t^3 + t^5)}{\cos^5\theta(1 - 10t^2 + 5t^4)}$

$= \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$

iv) Let $\theta = \frac{\pi}{5}$ so $t = \tan\frac{\pi}{5}$

and $\tan 5\theta = \tan\pi = 0$

so $0 = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$

$0 = t(t^4 - 10t^2 + 5)$

But $t = \tan\frac{\pi}{5} \neq 0$ so,

$(t^2)^2 - 10t^2 + 5 = 0$

$t^2 = \dfrac{10 \pm \sqrt{100 - 4 \times 5}}{2}$

$= 5 \pm 2\sqrt{5}$

but since $\tan x$ is always increasing, $\tan\frac{\pi}{5} < \tan\frac{\pi}{4}$

$$\tan\frac{\pi}{5} < 1$$

so $t^2 < 1$

$t^2 = 5 - 2\sqrt{5}$

$t = \pm\sqrt{5 - 2\sqrt{5}}$

but $t = \tan\frac{\pi}{5} > 0$

$\therefore \tan\frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$

Question 15

a) i)

$$\dot{y} = \frac{(10 + ku\sin\theta)}{k^2}(ke^{-kt}) - \frac{10}{k}$$

and max height is at $\dot{y} = 0$

$$\frac{(10 + ku\sin\theta)ke^{-kt}}{k^2} = \frac{10}{k}$$

$$e^{-kt} = \frac{10}{10 + ku\sin\theta}$$

$$-kt = \ln\left(\frac{10}{10 + ku\sin\theta}\right)$$

$$t = \frac{1}{k}\ln\left(\frac{10 + ku\sin\theta}{10}\right)$$

ii) $t = \frac{1}{0.5}\ln\left(\frac{10 + 0.5\times 40\times\sin 30^\circ}{10}\right)$

$= 2\ln 2$

at $t = 2\ln 2$:

$$y = \frac{10 + 0.5\times 40\times\sin 30^\circ}{0.5^2}\left(1 - e^{-0.5(2\ln 2)}\right) - \frac{10(2\ln 2)}{0.5}$$

$= 80\left(1 - \frac{1}{2}\right) - 40\ln 2$

$= 40 - 40\ln 2$

$= 40(1 - \ln 2)$

$$x = \frac{40\cos 30^\circ}{0.5}\left(1 - e^{-0.5(2\ln 2)}\right)$$

$= 40\sqrt{3}\left(1 - \frac{1}{2}\right)$

$= 20\sqrt{3}$

$\therefore$ max height at $(20\sqrt{3}, 40(1 - \ln 2))$

b) i) $I_n = \int_0^1 (1 + x^2)^{-n}\,dx$

$u = (1 + x^2)^{-n}$ $\quad v' = 1$

$u' = -n(1 + x^2)^{-n-1} \cdot 2x$ $\quad v = x$

$= \frac{-2nx}{(1 + x^2)^{n+1}}$

so, $I_n = \left[x(1 + x^2)^{-n}\right]_0^1 - \int_0^1 \frac{-2nx^2}{(1 + x^2)^{n+1}}\,dx$

$= \frac{1}{2^n} + 2n\int_0^1 \frac{1 + x^2 - 1}{(1 + x^2)^{n+1}}\,dx$

$= \frac{1}{2^n} + 2n\int_0^1 \frac{1}{(1 + x^2)^n}\,dx - 2n\int_0^1 \frac{dx}{(1 + x^2)^{n+1}}$

$I_n = \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$

$$2nI_{n+1} = 2nI_n - I_n + \frac{1}{2^n}$$

$$I_{n+1} = \frac{2nI_n - I_n}{2n} + \frac{1}{2n.2^n}$$

$$= \frac{2n-1}{2n} I_n + \frac{1}{(2^{n+1})n}$$

ii)
$$I_1 = \int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \left[\tan^{-1} x\right]_0^1$$

$$= \frac{\pi}{4}$$

$$I_2 = \frac{2-1}{2} \cdot \frac{\pi}{4} + \frac{1}{2^2}$$

$$= \frac{\pi+2}{8}$$

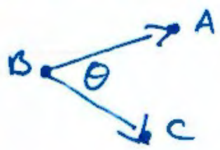
$$\int_0^1 \frac{1}{(1+x^2)^3}\,dx = I_3$$

$$= \frac{4-1}{4} \times \frac{\pi+2}{8} + \frac{1}{2^3 \times 2}$$

$$= \frac{3(\pi+2)}{32} + \frac{1}{16}$$

$$= \frac{3\pi+8}{32}$$

c) i) Let $\angle ABC = \theta$



$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}| \cos\theta$$

$$\overrightarrow{BA} = \begin{pmatrix} -1 \\ \frac{4}{3} \\ 7 \end{pmatrix} - \begin{pmatrix} 4 \\ \frac{4}{3} \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix}$$

$$|\overrightarrow{BA}| = \sqrt{(-5)^2 + 5^2} = \sqrt{50}$$

$$\overrightarrow{BC} = \begin{pmatrix} 6 \\ \frac{16}{3} \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ \frac{4}{3} \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

$$|\overrightarrow{BC}| = \sqrt{2^2 + 4^2} = \sqrt{20}$$

$$\begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \sqrt{50}\sqrt{20} \cos\theta$$

$$\cos\theta = \frac{-10}{\sqrt{1000}} = \frac{-1}{\sqrt{10}}$$

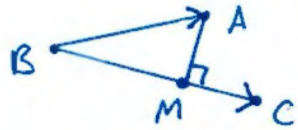
ii)

$$\sin\theta = \frac{3}{\sqrt{10}}$$

$$\text{Area } \triangle ABC = \frac{1}{2}\sqrt{50}\sqrt{20} \cdot \frac{3}{\sqrt{10}}$$

$$= 15 \text{ units}^2$$

iii)



$$\overrightarrow{BM} = \text{proj}_{\overrightarrow{BC}} \overrightarrow{BA}$$

$$= \frac{\begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}}{\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

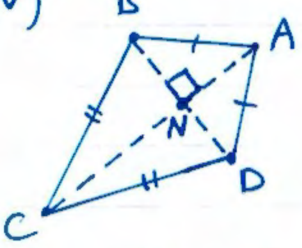
$$= \frac{-10}{20} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$$

$$|\overrightarrow{BM}| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$

$$AM = \sqrt{\sqrt{50}^2 - \sqrt{5}^2} \quad \text{(Pythagoras)}$$

$$= 3\sqrt{5}$$

iv)



$$\overrightarrow{BN} = \overrightarrow{BA} + \lambda \overrightarrow{AC}$$

$$= \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix} + \lambda \left[\begin{pmatrix} 6 \\ \frac{16}{3} \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ \frac{4}{3} \\ 7 \end{pmatrix} \right]$$

$$= \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} -5 + 7\lambda \\ 4\lambda \\ 5 - 5\lambda \end{pmatrix}$$

Also since $AC \perp BD$:

$$\overrightarrow{BN} \cdot \overrightarrow{AC} = 0$$

$$\begin{pmatrix} -5 + 7\lambda \\ 4\lambda \\ 5 - 5\lambda \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix} = 0$$

$$-35 + 49\lambda + 16\lambda - 25 + 25\lambda = 0$$

$$90\lambda = 60$$

$$\lambda = \frac{2}{3}$$

So, $\overrightarrow{BN} = \begin{pmatrix} -5 + \frac{14}{3} \\ \frac{8}{3} \\ 5 - \frac{10}{3} \end{pmatrix}$

$$= \frac{1}{3} \begin{pmatrix} -1 \\ 8 \\ 5 \end{pmatrix}$$

And $\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BN}$

$$= \begin{pmatrix} 4 \\ \frac{4}{3} \\ 2 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} -1 \\ 8 \\ 5 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 5 \\ 10 \\ 8 \end{pmatrix}$$

Question 16

a)

For $t = \tan\frac{x}{2}$ $\quad \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$$= \frac{1}{2}(1 + t^2)$$

$$dx = \frac{2\,dt}{1+t^2}$$

$x = 0$, $t = 0$ and $x = \frac{\pi}{2}$, $t = 1$

$$LHS = \int_0^1 \frac{1}{2 - \frac{1-t^2}{1+t^2} + \frac{4t}{1+t^2}} \times \frac{2\,dt}{1+t^2}$$

$$= \int_0^1 \frac{2}{2 + 2t^2 - 1 + t^2 + 4t}\,dt$$

$$= \int_0^1 \frac{2}{3t^2 + 4t + 1}\,dt$$

$$= \int_0^1 \frac{2}{(3t+1)(t+1)}\,dt$$

let $\frac{2}{(3t+1)(t+1)} = \frac{A}{3t+1} + \frac{B}{t+1}$

and $A = 3$, $B = -1$

$$LHS = \int_0^1 \left(\frac{3}{3t+1} - \frac{1}{t+1} \right) dt$$

$$= \Big[\ln(3t+1) - \ln(t+1) \Big]_0^1$$

$$= \ln 4 - \ln 2 - \ln 1 + \ln 1$$

$$= \ln 2$$

b) i) $\frac{n-1}{2}(1+(n-1))$

$= \frac{n(n-1)}{2}$

ii) Let $f(x) = x - \ln(1+x)$ for $x \geqslant 0$

$f(0) = 0 - \ln 1 = 0$

$f'(x) = 1 - \frac{1}{1+x}$

$= \frac{1+x-1}{1+x}$

$= \frac{x}{1+x}$

$> 0 \quad \forall x > 0$

So, $f(x)$ is increasing $\forall x > 0$
and since $f(0) = 0$, $f(x) > 0 \ \forall x > 0$
$\therefore x - \ln(1+x) > 0 \quad \forall x > 0$
$\therefore x > \ln(1+x) \quad \forall x > 0$

iii) Note:

$\binom{n}{2} = \frac{n!}{2(n-2)!} = \frac{n(n-1)(n-2)!}{2(n-2)!}$

$= \frac{n(n-1)}{2}$

$= 1+2+3+\ldots+(n-1)$ (by (i))

From part (ii):

$1 > \ln 2$
$2 > \ln 3$
$\vdots$
$n-1 > \ln n$

Adding together gives:

$1+2+3+\ldots+(n-1) > \ln 2 + \ln 3 + \ldots + \ln n$

$\binom{n}{2} > \ln(2 \times 3 \times 4 \times \ldots \times n)$

$\binom{n}{2} > \ln(n!)$

$e^{\binom{n}{2}} > e^{\ln(n!)}$ since both sides are positive

$\therefore e^{\binom{n}{2}} > n!$

c) i) Since $f(x)$ is the sum of squares of reals, $f(x) \geqslant 0 \quad \forall x \in \mathbb{R}$.

$\left(\sqrt{a_k}\,x - \frac{1}{\sqrt{a_k}}\right) = (\sqrt{a_k}\,x)^2 - 2\frac{\sqrt{a_k}\,x}{\sqrt{a_k}} + \left(\frac{1}{\sqrt{a_k}}\right)^2$

$= a_k x^2 - 2x + \frac{1}{a_k}$

$f(x) = \sum_{k=1}^{n}\left(a_k x^2 - 2x + \frac{1}{a_k}\right)$

$= (a_1 + a_2 + \ldots + a_n)x^2 - 2nx + \left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)$

Since $f(x) \geqslant 0$, $\Delta \leq 0$:

$(-2n)^2 - 4(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \leq 0$

$-4(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \leq -4n^2$

$(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \geqslant n^2$

ii) Let $a_1 = 1$, $a_2 = 2$ etc. (i.e. $a_n = n$)

$$\therefore (1 + 2 + \dots + n)\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) \geqslant n^2$$

$$\frac{n(1+n)}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) \geqslant n^2$$

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \geqslant \frac{2n^2}{n(1+n)}$$

$$\therefore \quad 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \geqslant \frac{2n}{n+1}$$